

Applications of zero forcing number to the minimum rank problem

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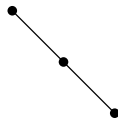
- Introduction and some related properties
- Exhaustive zero forcing number and sieving process
- Summary and a counterexample to a problem on edge spread

Relation between Matrices and Graphs

$\mathcal{G}$: real symmetric matrices $\rightarrow$ graphs.

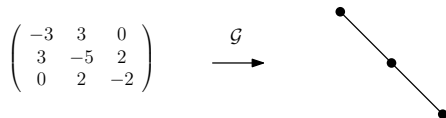
$$\begin{pmatrix} -3 & 3 & 0 \\ 3 & -5 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

$\xrightarrow{\mathcal{G}}$



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$$\mathcal{S}(G) = \{A \in M_{n \times n}(\mathbb{R}) : A = A^t, \mathcal{G}(A) = G\}.$$

- The **minimum rank** of a graph G is

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- The **minimum rank problem** of a graph G is to determine the number $\text{mr}(G)$ or $M(G)$.

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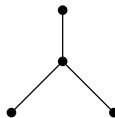
Related Parameters

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- The **path cover number** $P(G)$ of a graph G is the minimum number of vertex disjoint induced paths of G that cover $V(G)$.

Example for Three Parameters

$$\begin{pmatrix} ? & * & * & * \\ * & ? & 0 & 0 \\ * & 0 & ? & 0 \\ * & 0 & 0 & ? \end{pmatrix}$$

$\xrightarrow{\mathcal{G}}$

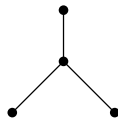


- $\text{rank} \geq 2$.

Example for Three Parameters

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\xrightarrow{\mathcal{G}}$

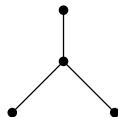


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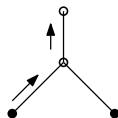


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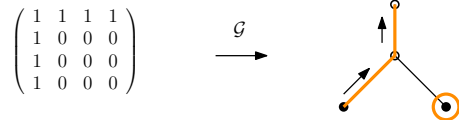
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Basic Properties

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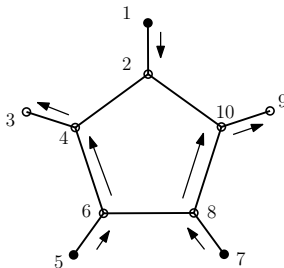
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- $M(G)$ and $P(G)$ are not comparable in general.

Terminologies for $Z(G)$

- A **chronological list** record the order of forces.



chronological list

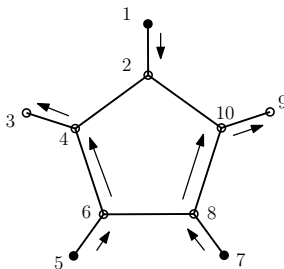
$1 \rightarrow 2$ $4 \rightarrow 3$
 $5 \rightarrow 6$ $8 \rightarrow 10$
 $7 \rightarrow 8$ $10 \rightarrow 9$
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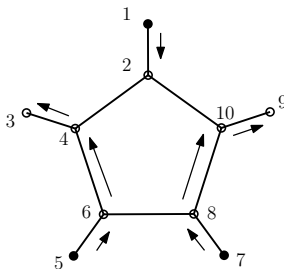
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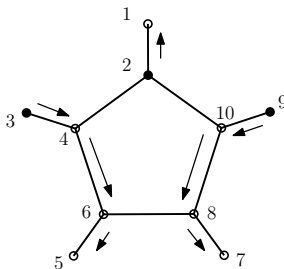
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- The inverse chronological list gives another zero forcing set called **reversal**.



chronological list

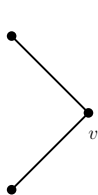
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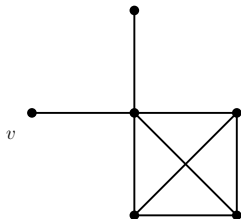
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Vertex-sum Operation

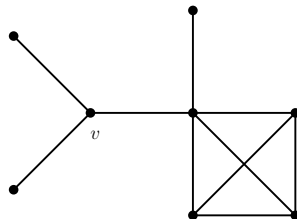
- The **vertex-sum** of G_1 and G_2 at the vertex v is the graph $G_1 \oplus_v G_2$ obtained by identifying the vertex v .



G_1



G_2

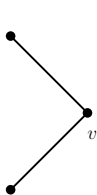


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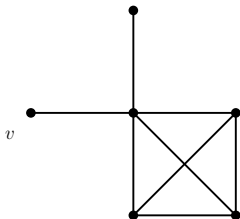
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- If $G = G_1 \oplus_v G_2$, then

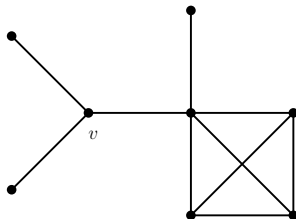
$$M(G) = \max\{M(G_1) + M(G_2) - 1, M(G_1 - v) + M(G_2 - v) - 1\}. [4]$$



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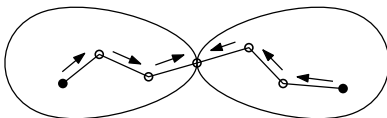
$$\left\{ \begin{array}{l} -1 \leq z_v(G) \leq 1. \\ v \text{ is doubly terminal} \Leftrightarrow z_v = 0. \\ v \text{ is simply terminal} \Rightarrow z_v = 0. \end{array} \right.$$

Sketch of Proof



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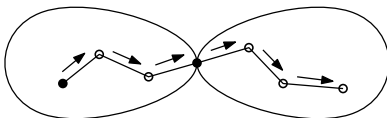


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- If $G = G_1 \oplus_v G_2$, then

$$Z(G) \leq Z(G_1) + Z(G_2 - v), \quad Z(G) \leq Z(G_1 - v) + Z(G_2),$$

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- $z_v(G) = -1, z_v(G_1) = z_v(G_2) = 0$ is the only possibility. This implies v is simply terminal for G_1 and G_2 .

Comparison of Reduction Formulae

- Denote $m_v(G) = M(G) - M(G - v)$,
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- $-1 \leq m_v, p_v, r_v \leq 1$.
- If $G = G_1 \oplus_v G_2$, they have similar behavior.

$m_v(G_1 \setminus G_2)$	-1	0	1
-1	-1	-1	-1
0	-1	-1	0
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,			
$p_v, z_v(G_1 \setminus G_2)$	-1	0	1
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- Hard to apply on induction.

The PZ condition

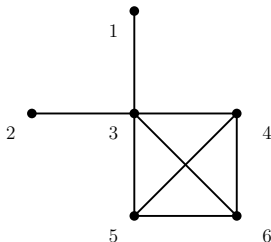
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- A graph G satisfies the **PZ condition** iff $P(G) = Z(G)$.

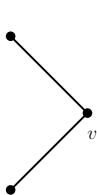
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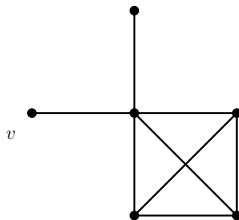


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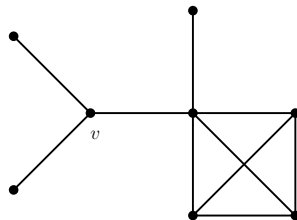
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- PZ condition does not preserve under vertex-sum operation.



G_1



G_2



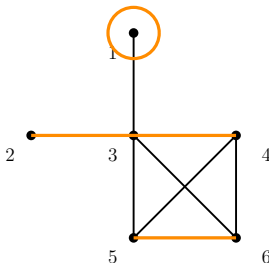
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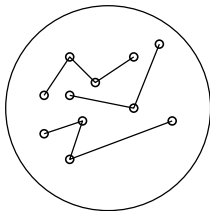
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- Strong PZ condition $\Rightarrow$ PZ condition.



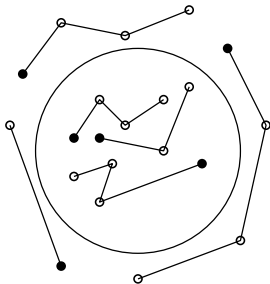
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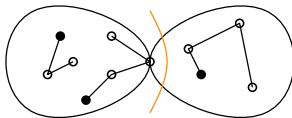
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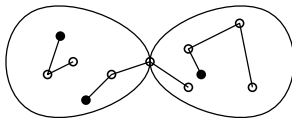
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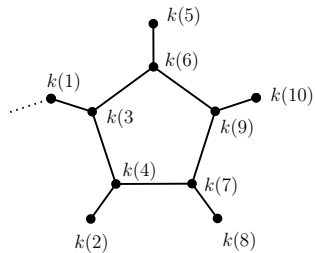
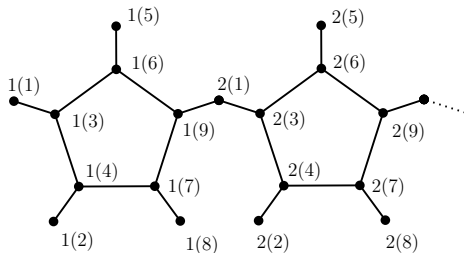


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- A cactus G satisfies the strong PZ condition. Hence we have $P(G) = Z(G)$.

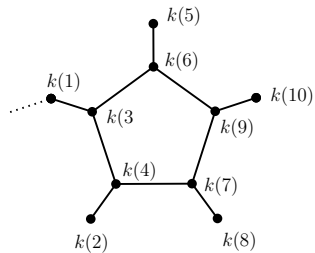
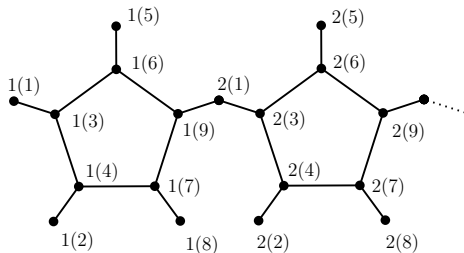
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- Let G_k be the k 5-sun sequence. Then
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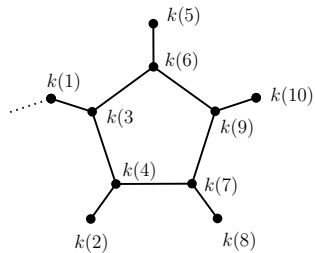
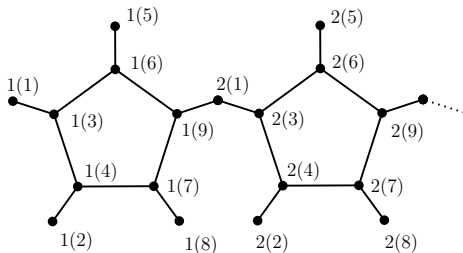
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- Q: Will the inequality $Z(G) \leq 2M(G) - 1$ holds for all G ?



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- The minimum rank of a pattern Q is

$$\text{mr}(Q) = \min\{\text{rank} A : A \cong Q\}.$$

Example for Minimum Rank of A Pattern

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- The rank 2 is achievable. Hence $\text{mr}(Q) = 2$.

Operation on S

- Define addition “+” and scalar multiplication “ $\times$ ” on S .

$$+: S \times S \rightarrow S$$

+	0	*	u
0	0	*	u
*	*	u	u
u	u	u	u

$$\times: \{0, *\} \times S \rightarrow S$$

$\times$	0	*	u
0	0	0	0
*	0	*	u

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- The **rank** of a pattern is the maximum number of independent **row** sign vectors.

Independence in different senses

Lemma

Suppose $V = \{v_1, v_2, \dots, v_n\}$ is a set of sign vectors, and $W = \{w_1, w_2, \dots, w_n\}$ is a set of sign vectors such that w_i is obtained from v_i by *replacing entries u by 0 or $*$* . If V is linearly independent, then so is W .

Suppose $R = \{r_1, r_2, \dots, r_n\}$ is a set of real vectors such that each entry in each vector *matches the corresponding entry* in elements of W . If W is linearly independent, then R is linearly independent as real vectors.

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Theorem

If Q is a pattern matrix and U is the set of all pattern matrices obtained from Q by **replacing u by 0 or $*$** , then

$$\text{rank}(Q) \leq \min_{Q' \in U} \{\text{rank}(Q')\} \leq \text{mr}(Q).$$

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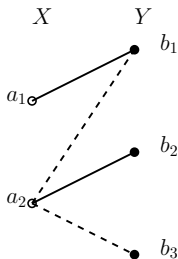
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- When W and B is empty, $Z_W(G, B) = Z(G)$.

Natural Relation between Patterns and Bipartites

- Q is a given $m \times n$ pattern. $G = (X \cup Y, E)$ is the related bipartite defined by

$$X = \{a_1, a_2, \dots, a_m\}, \quad Y = \{b_1, b_2, \dots, b_n\}, \quad E = \{a_i b_j : Q_{ij} \neq 0\}.$$

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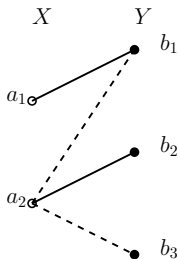
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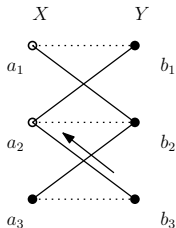
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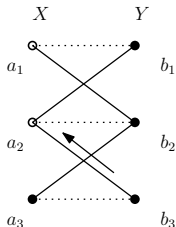
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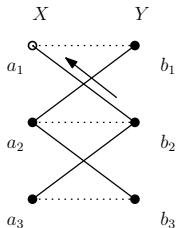
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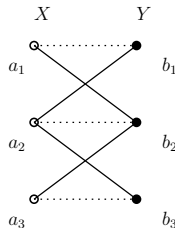
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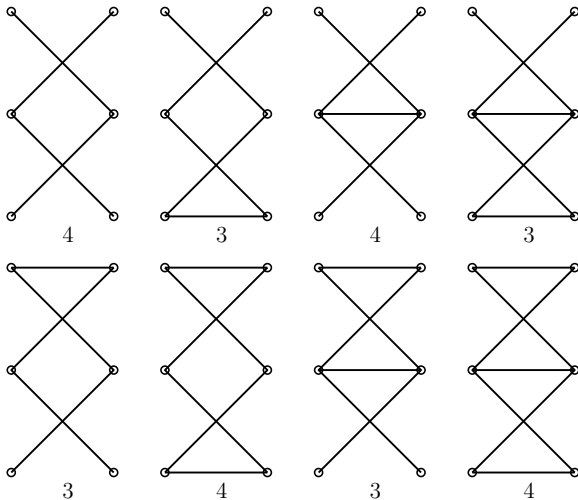
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Bipartites related to P_3



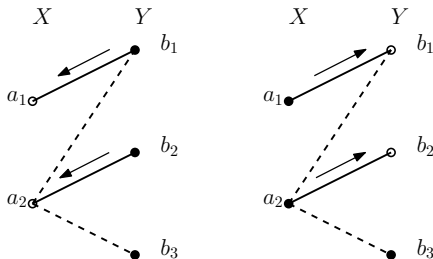
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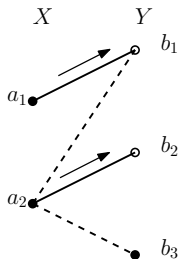
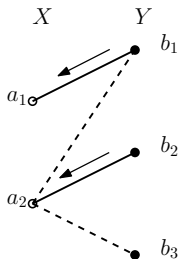
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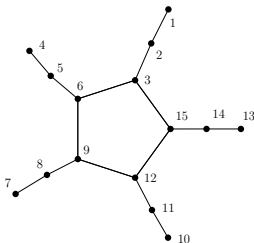
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- The parameter $\tilde{Z}(G)$ is still not sharp for some cactus.

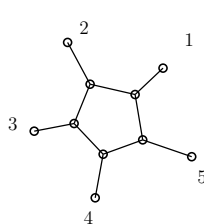
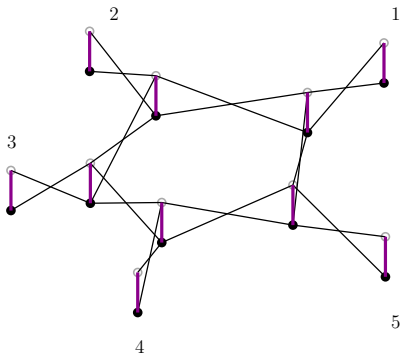


Example for Sieving Process

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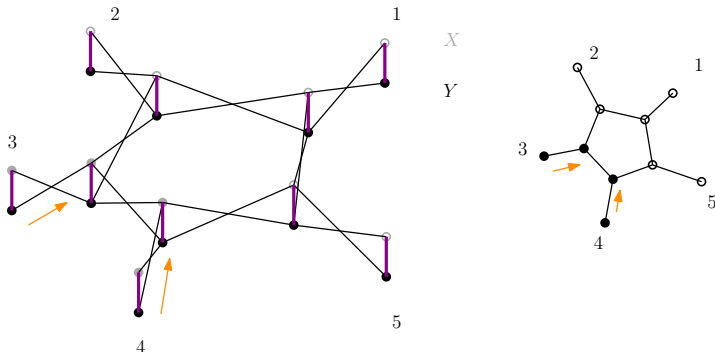


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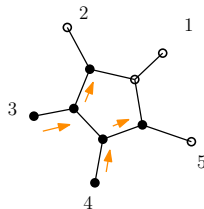
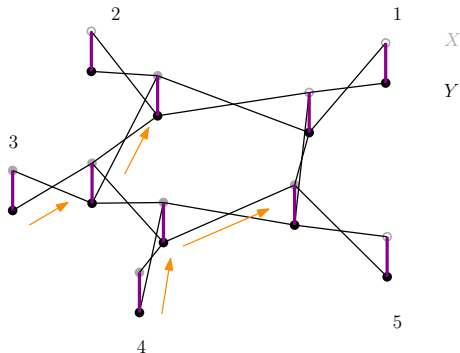


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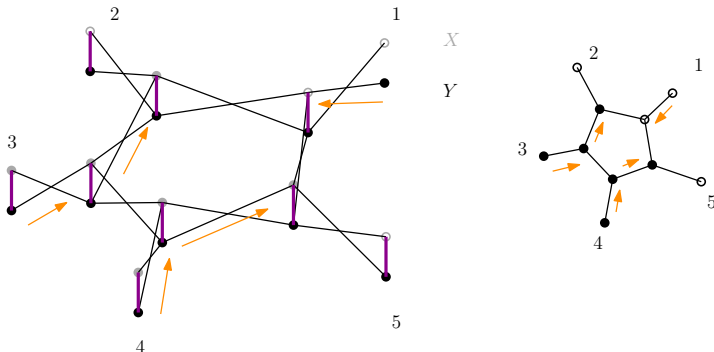


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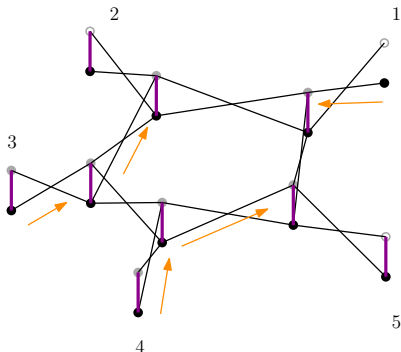


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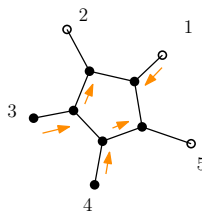
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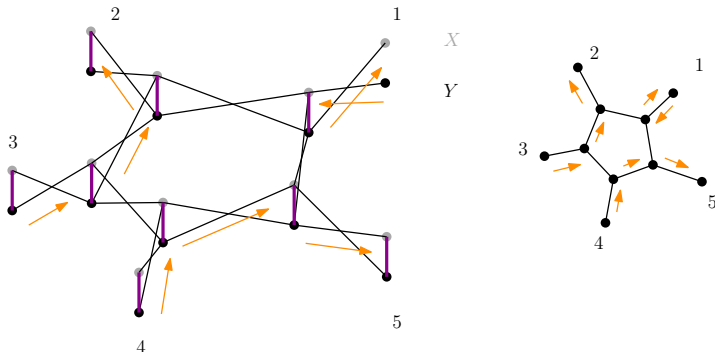


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- If $Z(\widetilde{H}_{5I}) - 10 = 3$ for some I , then $1 \in I$ and $2 \notin I$, a contradiction.

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$$\widetilde{Z}(G) = 12 - 10 = 2.$$

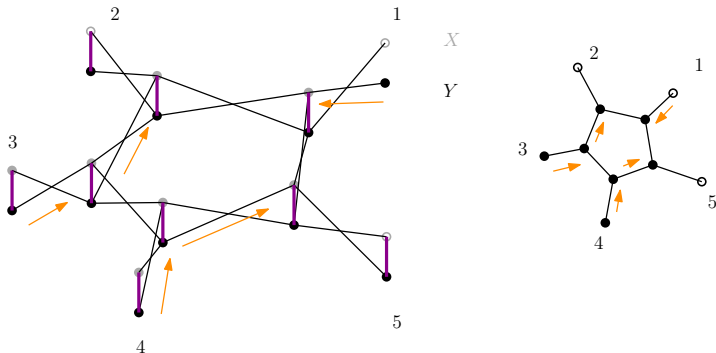


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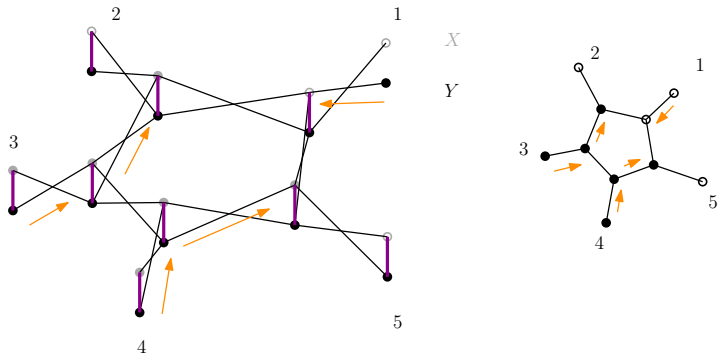


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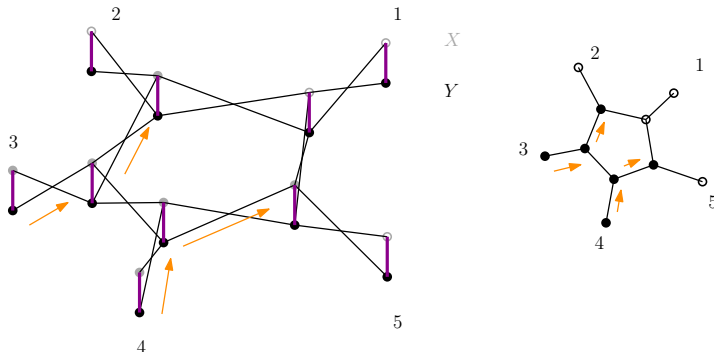


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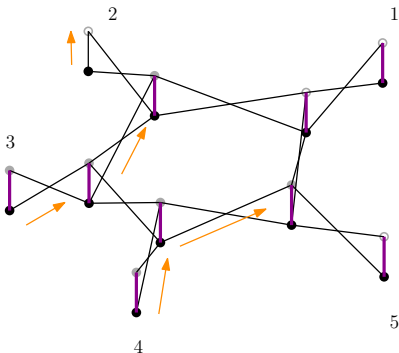


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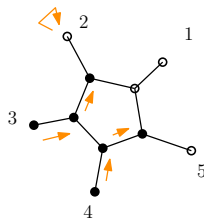
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X

Y

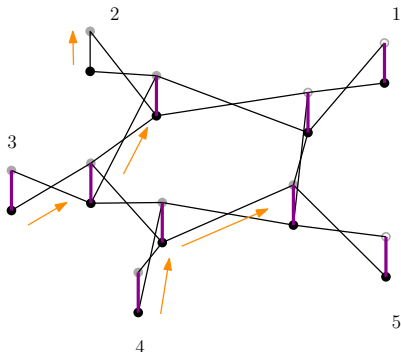


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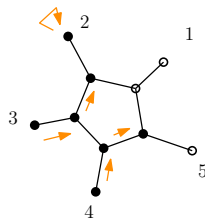
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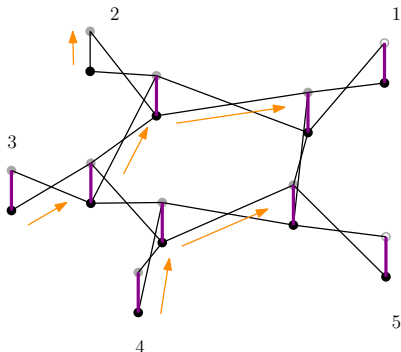


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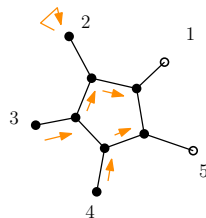
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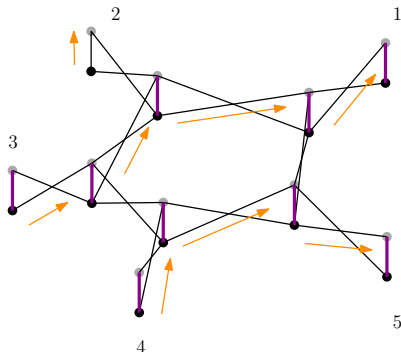


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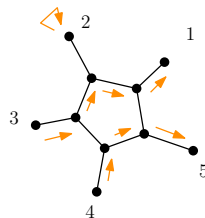
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Edge vs Nonedge

- Edge: **Increase** number of neighbor; **Increase** possible route for passing.

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- Each $F \supseteq Y$ with size $n + k - 1$ is a **sieve** for $\mathcal{I}_k(G)$ to delete impossible index sets.

Nonzero-vertex and Zero-vertex

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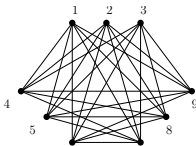
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- For multi-partite G with more than one part and more than two vertices in each parts, each vertex is a zero-vertex in $\mathcal{I}_{n-2}(G)$, $n = |V(G)|$.
- We know $Z(G_k) = 2k + 1$ and $M(G_k) = k + 1$. By sieving process, $\tilde{Z}(G_k) = k + 1$! Here G_k is the k 5-sun sequence.

Example for Stronger Upper Bound 1

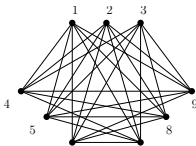
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$$A = \begin{pmatrix} O & J & J \\ J & O & B^t \\ J & B & O \end{pmatrix}.$$



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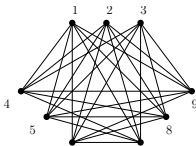
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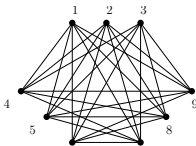
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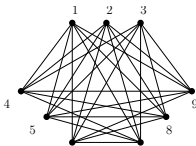
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- $M(G) \leq 6$. And actually $M(G) = 6$.



Theorem

For a graph G , suppose i is a **nonzero-vertex** in $\mathcal{I}_k(G)$. And $\eta_i(G)$ denote the set of those graphs obtained from G by the following rules:

- The vertex i should be deleted;
- For any neighbors x and y of i , the pair xy should be an edge if $xy \notin E(G)$ and could be an edge or a non-edge if $xy \in E(G)$.

If the nullity k is achievable by some matrix in $\mathcal{S}(G)$, then

$$k \leq \max\{M(H) : H \in \eta_i(G)\}.$$

- If k is achievable by $A \in \mathcal{S}(G)$, assume

$$A = \begin{pmatrix} 1 & a^t & 0 \\ a & \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{21} & \widehat{A}_{22} \end{pmatrix}.$$

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Zero Elimination Lemma

Theorem

For a graph G , suppose i is a **zero-vertex** in $\mathcal{I}_k(G)$ and j is a **neighbor of i** . Let

$$N_1 = \{v: iv \in E(G), v \neq j\}, \quad N_2 = \{v: jv \in E(G), iv \notin E(G), v \neq i\}.$$

And $\eta_{i \rightarrow j}(G)$ denote the set of those graphs obtained from G by the following rules:

- The vertex i and j should be deleted;
- For $x \in N_1$ and $y \in N_2$, the pair xy should be an edge if $xy \notin E(G)$ and could be an edge or a non-edge if $xy \in E(G)$;
- For x and y in N_1 , the pair xy could be an edge or a non-edge.

If the nullity k is achievable by some matrix in $\mathcal{S}(G)$, then

$$k \leq \max\{M(H): H \in \eta_{i \rightarrow j}(G)\}.$$

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- If k is achievable by $A \in \mathcal{S}(G)$, assume

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Here α has the form $\begin{pmatrix} 0 & * \\ * & u \end{pmatrix}$ and α^{-1} has the form $\begin{pmatrix} u & * \\ * & 0 \end{pmatrix}$.

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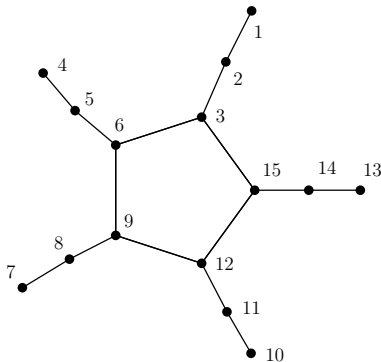
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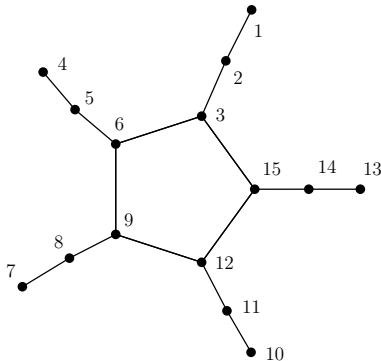
Example for Stronger Upper Bound 2

- $\tilde{Z}(G) = Z(G) = P(G) = 3$.



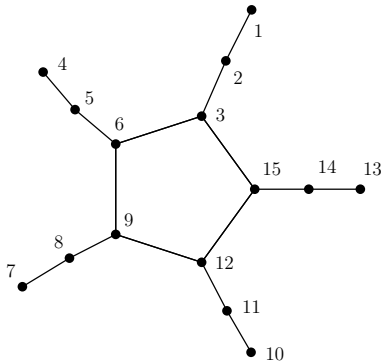
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- $\tilde{Z}(G) = Z(G) = P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $\mathcal{I}_3$.



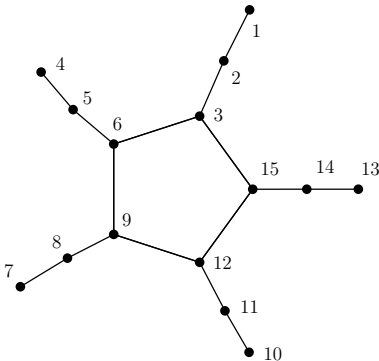
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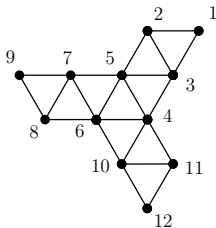
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Hence $M(G) \leq 2$.



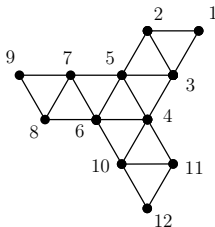
Example for Stronger Upper Bound 3

- $Z(G) = 4$ and $P(G) = 3$.



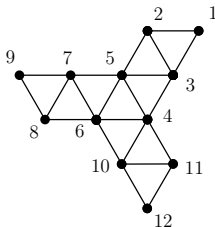
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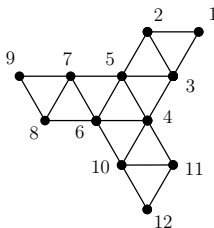
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- $Z(G) = 4$ and $P(G) = 3$.
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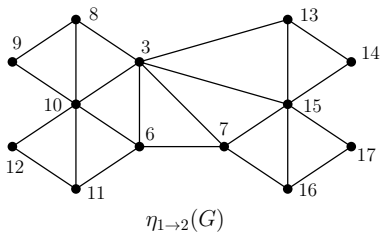
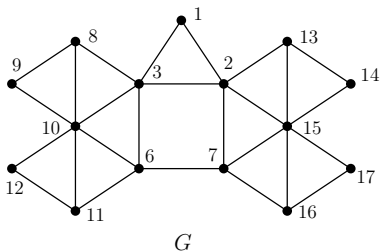
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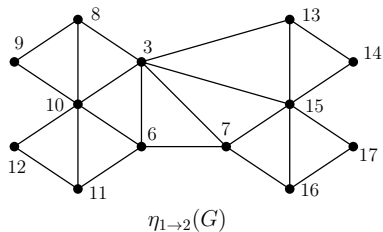
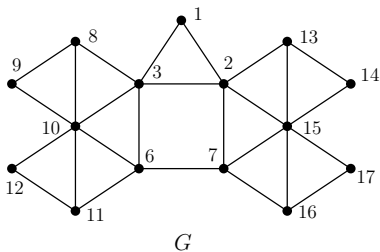
Example for Stronger Upper Bound 4

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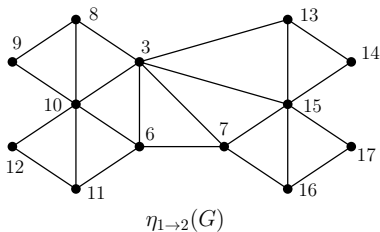
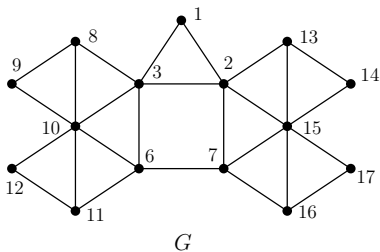
Example for Stronger Upper Bound 4

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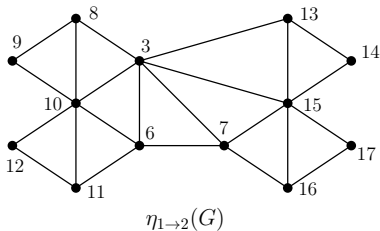
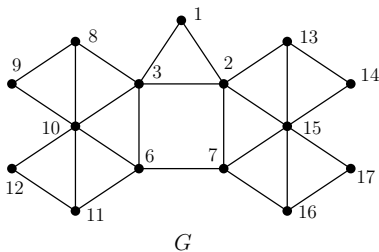
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- $\eta_{1 \rightarrow}(G)$ contains only one graph H .



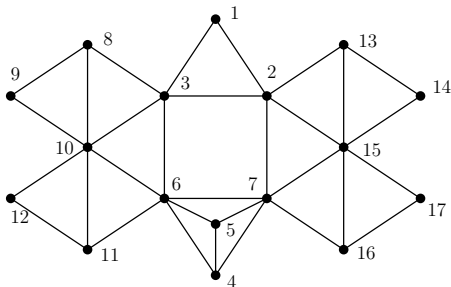
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- $\eta_{1 \rightarrow}(G)$ contains only one graph H .
- If 5 is achievable, then $5 \leq M(H) \leq 4$, a contradiction. Hence $M(G) \leq 4$.



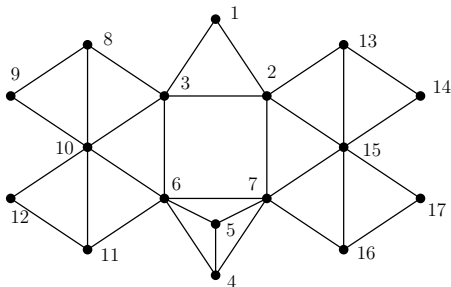
Example for Stronger Upper Bound 5

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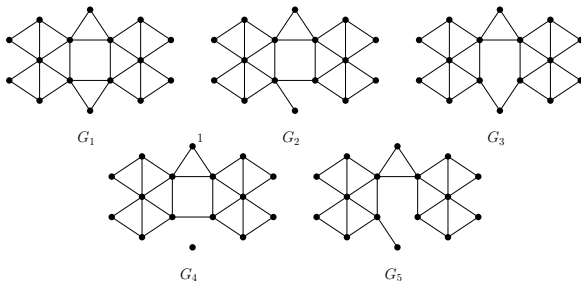
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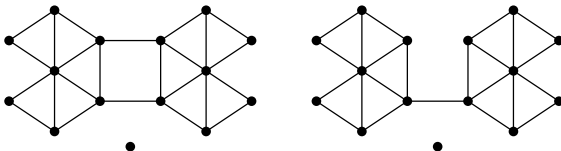
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Simple Elimination Lemma

Corollary

If i is a vertex of a graph G and j is a neighbor of i , then

$$M(G) \leq \max\{M(H) : H \in \eta_i(G) \cup \eta_{i \rightarrow j}(G)\}.$$

Enhanced Zero Forcing Number on Graph [9]

- A **looped graph** is a graph that allows loops. A vertex x is a neighbor of itself if and only if there is a loop on it.

Theorem

$\tilde{Z}(G) = \hat{Z}(G)$ for all graph G .

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Triangle Number on Pattern[3]

- A **t -triangle** of Q is a $t \times t$ subpattern that is permutation similar to a pattern that is upper triangular with all diagonal entries nonzero.

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- Question 2.22 in [7] ask whether the converse of Theorem 2.21 is true.

The Counterexample

- T is the turtle graph. $G = (X \cup Y, E)$ is construct from T by

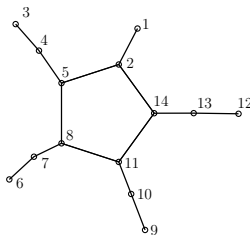
$$X = \{a_1, a_2, \dots, a_{14}\}, \quad Y = \{b_1, b_2, \dots, b_{14}\},$$

and

$$E(G) = E_1 \cup E_2,$$

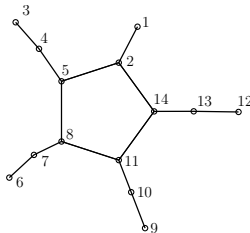
where

$$E_1 = \{a_i a_j : i \neq j\} \cup \{b_i b_j : i \neq j\}, \quad E_2 = \{a_i b_j : ij \in E(T) \text{ or } i = j\}.$$



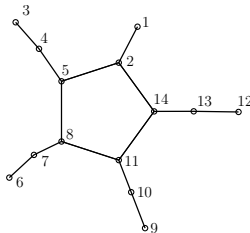
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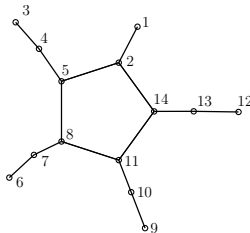
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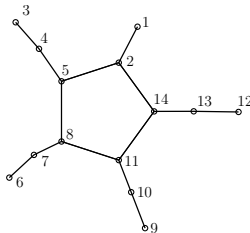
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- The edge $e = a_1b_1$ is used in each optimal zero forcing set. But $Z(G) = Z(G - e) = 16$ and so $z_e(G) = 0 \neq -1$.



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- The proof in [13] of $M(C_n) = 2$ could be generalized.
- $\text{mr}(G) = \text{mrs}(Q(G)) = \min\{\text{mrs}(Q_I(G))\}$. So it is still valuable to consider zero-nonzero symmetric min rank problem.

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